Discrete Mathematics Recitation Course 3

2013.04.11

張玟翔

3-1

Algorithms

3-1 Ex.2

- Determine which characteristics of an algorithm the following procedures they lack.
  - a) **procedure** *double*(n: positive integer)

    while $n > 0$

    $n := 2n$
  - c) **procedure** *sum*(n: positive integer)

    $sum := 0$

    while $i < 10$

    $sum := sum + i$

- This procedure is not finite, since **while** loop continues forever.
- This procedure lacks definiteness, since the value of i is never set.

3-1 Ex.8

- Describe an algorithm that takes as input a list of n distinct integers and finds the location of the largest even integer in the list or returns 0 if there are no even integers in the list.
- **procedure** *largest even location*($a_1, a_2, \dots, a_n$: integers)

$k := 0$

$largest := -\infty$

for $i := 1$ **to** n

  if (a_i is even and $a_i > largest$) **then**

    begin

      $k := i$

      $largest := a_i$

    end

end {k is the desired location (or 0 if there are no evens)}

The Bubble Sort

procedure *bubble sort*$(a_1, a_2, \ldots, a_n$: real numbers with $n \geq 2)$

for $i := 1$ **to** $n - 1$

  for $j := 1$ **to** $n - i$

    if $a_j > a_{j+1}$ **then** interchange a_j and a_{j+1}

$\{a_1, a_2, \ldots, a_n$ is in increasing order$\}$

3-1 Ex.34

- Use the bubble sort to sort 6, 2, 3, 1, 5, 4, showing the lists obtained at each step.

- 2,3,1,5,4,6
- 2,1,3,4,5,6
- 1,2,3,4,5,6
- 1,2,3,4,5,6
- 1,2,3,4,5,6

The Insertion Sort

procedure *insertion sort*($a_1, a_2, \ldots, a_n$: real numbers with $n \geq 2$)

for $j := 2$ **to** n

begin

$i := 1$

while $a_j > a_i$

$i := i + 1$

$m := a_j$

for $k := 0$ **to** $j - i - 1$

$a_{j-k} := a_{j-k-1}$

$a_i := m$

end $\{a_1, a_2, \ldots, a_n$ are sorted$\}$

3-1 Ex.38

- Use the insertion sort to sort 6, 2, 3, 1, 5, 4, showing the lists obtained at each step.

- 2,6,3,1,5,4
- 2,3,6,1,5,4
- 1,2,3,6,5,4
- 1,2,3,5,6,4
- 1,2,3,4,5,6

3-2

The Growth of Functions

3-2 Ex.4

- Use the definition of "$f(x)$ is $O(g(x))$" to show that $2^x + 17$ is $O(3^x)$.

- If $x > 5$, then $2^x + 17 \leq 2^x + 2^x = 2 \cdot 2^x \leq 2 \cdot 3^x$.
- By definition, $\exists c, k: \forall x > k: f(x) \leq cg(x)$.
- We can find a solution : $c = 2, k = 5$.

3-2 Ex.8

- Find the least integer n such that $f(x)$ is $O(x^n)$ for each of these functions.
  - a) $f(x) = 2x^2 + x^3\log x$ 4
  - b) $f(x) = 3x^5 + (\log x)^4$ 5
  - c) $f(x) = (x^4 + x^2 + 1)/(x^4 + 1)$ 0
  - d) $f(x) = (x^3 + 5\log x)/(x^4 + 1)$ -1

3-2 Ex.20

- Give a big-*O* estimate for each of these functions. For the function g in your estimate $f(x)$ is $O(g)$, use a simple function g of smallest order.
  - c) $(n^n + n2^n + 5^n)(n! + 5^n)$

- The dominant terms in the two factors are n^n and $n!$, respectively.
- Therefore this is $O(n^n n!)$.

3-2 Ex.22

- For each of these functions, determine whether that is $\Omega(x)$ and whether it is $\Theta(x)$.
  - a) $f(x) = 10$ not $\Omega(x)$, not $\Theta(x)$
  - b) $f(x) = 3x + 7$ $\Omega(x)$, $\Theta(x)$
  - c) $f(x) = x^2 + x + 1$ $\Omega(x)$, not $\Theta(x)$
  - d) $f(x) = 5\log x$ not $\Omega(x)$, not $\Theta(x)$
  - e) $f(x) = \lfloor x \rfloor$ $\Omega(x)$, $\Theta(x)$
  - f) $f(x) = \lceil x/2 \rceil$ $\Omega(x)$, $\Theta(x)$

3-2 Ex.36

- Suppose that $f(x)$ is $O(g(x))$. Does it follow that $2^{f(x)}$ is $O(2^{g(x)})$?

- This does not follow.
- Let $f(x) = 2x$ and $g(x) = x$, then $f(x)$ is $O(g(x))$.
- Now$2^{f(x)} = 2^{2x} = 4^x$, and$2^{g(x)} = 2^x$, so the ratio grows without bound as *x* grows.

3-3

Complexity of Algorithms

3-3 Ex.26

- Show that the greedy algorithm for making change for n cents using quarters, dimes, nickels, and pennies has $O(n)$ complexity measured in terms of comparisons needed.

- Each iteration (determining whether we can use a coin of a given denomination) takes a bounded amount of time, and there are at most n iterations , since each iteration decreases the number if cents remaining.
- Therefore there are $O(n)$ comparisons.

3-4

The Integers and Division

3-4 Ex.8

- Prove or disprove that if $a|bc$, where a, b, and c are positive integers, then $a|b$ or $a|c$.

- The simplest counterexample is provided by $a = 4$ and $b = c = 2$.

3-4 Ex.20

- Show that if $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$, where a, b, c, d, and m are integers with $m \geq 2$, then $a - c \equiv b - d \pmod{m}$.

- $a = b + sm$
- $c = d + tm$
- $a - c = (b - d) + (s - t)m$
- $a - c \equiv b - d \pmod{m}$

3-4 Ex.24

- Prove that if n is an odd positive integer, then $n^2 \equiv 1 \pmod 8$.

- Write $n = 2k + 1$ for some integer k.
- Then $n^2 = (2k + 1)\ 2 = 4k^2 + 4k + 1 = 4k(k + 1) + 1$.
- Since either k or $k + 1$ is even, $4k(k + 1)$ is a multiple of 8, therefore$n^2 - 1$ is a multiple of 8, so $n^2 \equiv 1 \pmod 8$.

3-5

Primes and Greatest Common Divisors

3-5 Ex.14

- We call a positive integer perfect if it equals the sum of its positive divisors other than itself.
  - (a) Show that 6 and 28 are perfect.
  - (b) Show that $2^{p-1}(2^p - 1)$is a perfect number when $2^p - 1$ is prime.

- 6=1+2+3; 28=1+2+4+7+14.
- Certainly all the numbers $1, 2, 4, 8, \ldots 2^{p-1}$ are proper divisors, and their sum is $2^p - 1$ (this is geometric series).
- Also each of these divisors times $2^p - 1$ is also a divisor, and all but the last is proper. Again adding up this geometric series we find a sum of $(2^p - 1)(2^{p-1} - 1)$.
- The sum of all the divisor is $(2^p-1) + (2^p - 1)(2^{p-1} - 1) = 2^{p-1}\ (2^p - 1)$, which is the original number.

3-5 Ex.26

- If the product of two integers is $2^7 3^8 5^2 7^{11}$ and their greatest common divisor is $2^3 3^4 5$, what is their least common multiple?

- The product of two integers = GCD*LCM.
- $(2^7 3^8 5^2 7^{11})/(2^3 3^4 5) = 2^4 3^4 5^1 7^{11}$.